

## Homework #2

### Math 465/565

A few notes about this homework.

- You will be graded on your presentation as well as the content of the work. This means that all of your plots should have axis labels, a title and legends, and any other annotation that makes it clear what you are demonstrating.
- You may submit your homework to BlackBoard. Please submit a PDF of your homework.
- Students in 465 may do the problems for 565 for extra credit.

1. Find the rate of convergence of the sequence

$$s_n = \frac{\sin(n)}{n} \tag{1}$$

as  $n \rightarrow \infty$ .

2. Show that the sequence

$$s_n = \frac{1}{n^2} \tag{2}$$

converges *linearly*.

3. Compute the limit and determine the corresponding rate of convergence for the function

$$\lim_{x \rightarrow 0} \frac{e^x - \cos x - x}{x^2} \tag{3}$$

4. The following Matlab code produces the output shown below

```
x = 1;
for k = 1:1074
    x = x/2;
    fprintf('%-7d %20.16e\n',k, x);
end
fprintf('\n');
fprintf('%10s %16.8e\n', 'realmin', realmin);
fprintf('%10s %16.8e\n', 'x', x);
```

Output

```
1      5.0000000000000000e-01
2      2.5000000000000000e-01
3      1.2500000000000000e-01
4      6.2500000000000000e-02
.....
1071   3.9525251667299724e-323
1072   1.9762625833649862e-323
1073   9.8813129168249309e-324
1074   4.9406564584124654e-324

realmin  2.22507386e-308
x        4.94065646e-324
```

For the following, you are asked to show both the *binary* and 64-bit representation of the requested values. The binary representation should be of the form

$$(1 + b_1 2^{-1} + b_2 2^{-2} + b_3 2^{-3} + \dots) \times 2^\beta \quad (4)$$

where  $b_i = 0$  or  $1$ , and  $\beta$  is an integer exponent. The 64-bit representation is a string of 64 0s and 1s.

- What is the value of iterate 52? Show the binary and 64-bit representations of this iterate. What is the name we often give to this number?
- Which iterate produces the value `realmin`? Show the binary and 64-bit representation of this number.
- Show the binary and 64-bit representations for the last value of  $x$  shown, and be sure to mention any IEEE floating point conventions that are involved. What is the name we give to these values smaller than `realmin`?
- Had the loop continued one more iteration, what would the resulting value of  $x$  be? Explain why.
- Using the Float-Toy (<http://evanw.github.io/float-toy/>), give an example of a number less than `realmin` that is not generated by the iterations above.

5. In class, we showed that

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - 1}{x^2} = \frac{1}{2} \quad (5)$$

as  $x \rightarrow 0$ . We then argued that

$$\left| f(x) - \frac{1}{2} \right| \leq \frac{1}{8} x^2 \quad (6)$$

for  $x$  close to 0, and concluded that the "rate of convergence" of  $f(x)$  is  $\mathcal{O}(x^2)$ . To get to this conclusion, however, we simply disregarded all higher order terms in the Taylor Series expansion of  $f(x)$ , without justification.

Do the following to convince yourself, using graphic and analytic means, that the conclusion is justified.

- Provide convincing graphical evidence that  $f(x)$  converges like  $\mathcal{O}(x^2)$  as  $x \rightarrow 0$ .
  - Plot  $|f(x) - 1/2|$  and  $x^2/8$  on the same graph to see the convergence behavior of  $f(x)$ .
  - Use small enough values of  $x$  to really see the behavior near 0. In Matlab, for example, use
 

```
x = logspace(-3,-1,500)
```

 This creates a sequence of values  $x_i$  that are equally spaced in a logarithmic scale. That is,  $x_i = 10^{p_i}$ , where the  $p_1 = -3$ ,  $p_{500} = -1$  and the remaining  $p_i$  are equally spaced values between  $-3$  and  $-1$ .
  - To see the behavior clearly, use a log scale for the y-axis. In Matlab, you can set the scale of the y-axis using
 

```
set(gca,'yscale','log')
```
- (Math 565.)** Use the Remainder Theorem to show formally that we are justified in claiming that the rate of convergence of  $f(x)$  is  $\mathcal{O}(x^2)$ . Is our choice of  $\lambda = 1/8$  reasonable?

6. Recreate the plot you created for Problem 5, but this time use very small values of  $x$ , i.e.

```
x = logspace(-6,-3,500)
```

You should see very strange behavior for  $x$  less than about  $10^{-4}$ . This behavior is the result of "catastrophic cancellation". To understand this, consider the following expression

$$g(\varepsilon) = \sqrt{1 + \varepsilon^2}, \quad \varepsilon \ll 1 \quad (7)$$

- (a) Show in finite precision arithmetic (roughly 16 decimal digits), we are justified in approximating  $g(\varepsilon)$  as

$$g(\varepsilon) \approx 1 + \frac{1}{2}\varepsilon^2 \quad (8)$$

for  $\varepsilon \lesssim 10^{-4}$ . **Hint:** What is the size of the next term in the Taylor Series for  $g(\varepsilon)$ ?

- (b) Show the result of each operation needed to evaluate  $f(\varepsilon)$ , given by

$$f(\varepsilon) \approx \frac{(1 + \frac{1}{2}\varepsilon^2) - 1}{\varepsilon^2} \quad (9)$$

in finite precision arithmetic, for  $\varepsilon = 10^{-4}$ . Explain how "garbage" digits can appear in the calculation, resulting in the strange behavior you see in your plot of  $f(x)$ .

- (c) For what value of  $\varepsilon$  do you expect to see almost no digits of accuracy in your evaluation of  $f(\varepsilon)$ ?  
(d) Recreate the plot from Problem 5 for these very small values, but this time use the Taylor Series approximation to  $f(x)$ . Include terms up to  $x^6$ .

7. (**Math 565**). Consider the sequence of partial sums

$$S_n(x) = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n = \sum_{k=0}^n (-1)^k x^k \quad (10)$$

for  $|x| < 1$ .

- (a) Numerically generate values of  $S_n(x)$  for  $x = 1/\pi$ . Show that the error sequence

$$e_n(x) = \left| S_n(x) - \frac{1}{1+x} \right| \quad (11)$$

converges linearly with asymptotic error constant  $x$ . Turn in your numerically generated sequence of values, and convincing numerical evidence that the sequence converges linearly.

- (b) Show analytically that the sequence  $S_n(x)$  converges linearly.