Homework # 6

Math 427/527

Note : Math 427 students may do the Math 527 questions for extra credit. You may work in pairs on this assignment, but pairs can only be two 427 students or two 527 students but not mixed pairs.

If you work together (pairs of two only), *you may turn in a single homework with both names.*

All plots must have axes labels, and a title. Also, be sure to use appropriate axis limits for each plot. Make your plots interesting!

1. Develop a Fourier-Legendre series

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x)$$

for the functions 1, x, x^2, x^3, and x^4. Show the details of your work.

2. Often, we can use knowledge of integral transforms to evaluate integrals. If we recognize that an integral is actually the transform of a known function, we can determine the value of that integral. Use this idea to show that

$$\int_0^{\infty} \frac{w^3 \sin xw}{w^4 + 4}\, dw = \frac{1}{2}\pi e^{-x} \cos x, \qquad x > 0$$

If you use Wolfram to compute any relevant integrals, be sure to indicate exactly which integral(s) you evaluated. Note : Do *not* just evaluate the integral above!

3. We have said that only *absolutely integrable* functions have Fourier transforms. This isn't entirely correct. For example, typing "Fourier Transform sin(x)" into Wolfram Alpha[1] returns

$$\mathcal{F}[\sin x] = \hat{f}(w) = i\sqrt{\frac{\pi}{2}}\left(\delta(w+1) - \delta(w-1)\right)$$

where $\delta(x)$ is the Dirac Delta function. Use the "sifting" property of the Dirac function

$$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(\xi - x)\, d\xi$$

to verify that

$$\mathcal{F}^{-1}[\hat{f}(w)] = \sin x$$

4. The following questions refer to Table III on Fourier Transforms from the course text.

   (a) Assume that $f(x)$ has a Fourier transform. Show that $\mathcal{F}[f(x-a)] = e^{-iwa}\mathcal{F}[f(x)]$.

   (b) Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w-a)$ is the Fourier transform of $e^{iax}f(x)$.

   (c) Use (4a) to obtain entry (1) from entry (2) in Table III.

   (d) Use (4b) to obtain entry (7) from entry (1) and entry (8) from entry (2) in Table III.

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[1] Actually, this is what Wolfram Alpha returns for the inverse Fourier Transform. WA reverses the sign on the argument to the complex exponential.