Homework #2

Math 427/527

Note : Math 427 students may do the Math 527 questions for extra credit. You may work in pairs on this assignment, but pairs can only be two 427 students or two 527 students but not mixed pairs.

All plots must have axes labels, and a title. Also, be sure to use appropriate axis limits for each plot. Make your plots interesting!

1. Show that for non-integer values of ν, the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are independent.

2. This problem will show you an interesting property of the function $\Gamma(x)$. Create a plot over integer values $0, 1, 2, \ldots, 10$ of the function f(n) = $n!$, where $n!$ is "n-factorial". Now, on the same graph, plot the continuous function $\Gamma(x+1)$ over the domain $x \in [0, 10]$. You should see that the gamma function interpolates the factorial values. To demonstrate this property, print a table of values comparing $\Gamma(n+1)$ and $n!$ for integer values $n = 0, 1, 2, \ldots, 10$, and conclude that $\Gamma(n) = (n-1)!$ (at least for small n). Hint : For the plot, use the `semilogy` function. For example, to plot the points $(0, 0!), (5, 5!)$ and $(10, 10!)$, you could use:

   ```
   semilogy([0:5:10],[factorial(0:5:10)],'.','markersize',30);
   ```

   Try "`help colon`" at the MATLAB prompt if you are unfamilar with the colon (":") operator.

3. Bessel functions come in two pairs of independent functions $(J_\nu(x), Y_\nu(x))$ and $(I_\nu(x), K_\nu(x))$, where $J_\nu(x)$ and $I_\nu(x)$ are "first kind" functions and $Y_\nu(x)$ and $K_\nu(x)$ are "second kind" functions. If you do a search on Bessel functions in Wikipedia, you will come across some intimidating formulas. We will de-mystify a few here. **Note** : We use ν in place of the α used by Wikipedia.

   (a) The Modified Bessel function of the first kind of order ν is defined by $I_\nu(x) = i^{-\nu} J_\nu(ix)$, where $i = \sqrt{-1}$. Show that I_v satisfies the ODE

   $$x^2 y'' + xy' - (x^2 + \nu^2)y = 0$$

   (b) Show that for $\nu = 0$, the following is exactly the first independent solution to Problem 3a that we found in class.

   $$I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

   For a_0, use the choice

   $$a_0 = \frac{1}{2^n n!}$$

   (c) The conventional choice of a second independent solution to the ODE from Problem 3a is expressed using the formula

   $$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$$

   for any ν. Show that $K_\nu(x)$ satisfies the ODE in Problem 3a.

   (d) Show that the formula in Figure 1, taken from Wikipedia is equivalent to the series solution for $Y_n(x)$ given by expression (8) in Section 5.5 of the class textbook.

4. In class, we found the first solution to the differential equation.

   $$xy'' + y' - xy = 0$$

   Find a second independent solution as a series solution using the Method of Frobenius. This solution is related to the Modified Bessel function of the second kind. Show all of your work.

$$Y_n(z) = -\frac{\left(\frac{z}{2}\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z^2}{4}\right)^k + \frac{2}{\pi}J_n(z)\ln\frac{z}{2} - \frac{\left(\frac{z}{2}\right)^n}{\pi}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(n+k+1))\frac{\left(-\frac{z^2}{4}\right)^k}{k!(n+k)!}$$

Figure 1: Formula from Wikipedia entry "Bessel function", used in Problem 3d.

5. Solve the ODE in Problem 4 numerically, using `ode45` over the domain $x \in [0.5, 5]$. Take as your boundary conditions $y(0.5) = y'(0.5) = 1$. Plot your solution. On the same graph, plot the exact solution (obtained using WolframAlpha, for example) over the domain $[0, 5]$. Describe an advantage of having the analytic solution, rather than just a numerical solution for this problem.

6. (**Math 527**) The classic text "Conduction of Heat in Solids", by H. S. Carslaw and J. C. Jaeger (Oxford University Press, 1959) proposes a model for heat flow in a wire. The model geometry is a cylinder of radius a and the distribution of temperature $T(r)$ in the wire is a function of radius r only. Assuming constant thermal resistivity $R = 1/K$, a simple model of heat flow in the wire is given by

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{A_0}{K} = 0, \qquad 0 \le r \le a \tag{1}$$

where K is the thermal conductivity of the wire and A_0 is constant rate of heat production due to Joule heating.

(a) A more realistic model allows the thermal resistivity R to vary linearly with temperature as $R = R_0(1 + \alpha(T - T_0)$, where R_0 is the resistance at a reference temperature T_0 and α is the temperature coefficient of resistivity. Show that the model in (1) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \beta^2 T = -\frac{A_0}{K_0}(1 - \alpha T_0), \qquad \beta^2 = \alpha A_0/K_0, \tag{2}$$

where $K_0 = 1/R_0$.

(b) Find the general solution to this model. **Hint 1** : First solve the homogeneous problem, then find a very simple solution to the non-homogeneous problem. Use the principle of superposition to find the general solution. **Hint 2** : Be sure your solution is physical over the domain $0 \le r \le a$.

(c) Suppose the temperature on the surface of the wire is held fixed at $T(a) = T_0$. Find the particular solution to model equation.

(d) Find physical values for resistivity R_0, rate of heat production, temperature coefficient α at a reference temperature T_0, diameter, and a surface temperature T_0 for copper wiring. Verify that the units you use are consistent. Plot your solution $T(r)$ using these values. Cite the sources you used.

Hint : For the units part of this question, convince yourself that the equation makes sense when K_0 is a thermal conductivity. Then to convert between thermal conductivity and electrical conductivity of a metal, use the Wiedemann-Franz Law, which states that

$$K_0 = LT_0\sigma$$

where σ is the electical conductivity ($\Omega^{-1}\,\mathrm{m}^{-1}$) and $L = 2.44 \times 10^{-8}$ is the Lorenz number ($\mathrm{W}\,\Omega\,\mathrm{K}^{-2}$).